

On Vanishing Fermat Quotients and a Bound of the Ihara Sum

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Abstract

We improve an estimate of A. Granville (1987) on the number of vanishing Fermat quotients $q_p(\ell)$ modulo a prime p when ℓ runs through primes $\ell \leq N$. We use this bound to obtain an unconditional improvement of the conditional (under the Generalised Riemann Hypothesis) estimate of Y. Ihara (2006) on a certain sum, related to vanishing Fermat quotients. In turn this sum appears in the study of the index of certain subfields of cyclotomic fields $\mathbb{Q}(\exp(2\pi i/p^2))$.

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1 Introduction

For a prime p and an integer u with $\gcd(u, p) = 1$ we define the *Fermat quotient* $q_p(u)$ as the unique integer with

$$q_p(u) \equiv \frac{u^{p-1} - 1}{p} \pmod{p}, \quad 0 \leq q_p(u) \leq p-1.$$

We also define $q_p(u) = 0$ for $u \equiv 0 \pmod{p}$.

Fermat quotients appear and play a major role in various questions of computational and algebraic number theory and thus have been studied in

a number of works, see, for example, [1, 2, 3, 5, 6, 7, 9, 10] and references therein. Amongst other properties, the p -divisibility of Fermat quotients $q_p(a)$ by p is important for many applications and in particular, the smallest value ℓ_p of $u \geq 1$ with $q_p(u) \neq 0$, has been studied in a number of works, see [1, 2, 3, 5, 9]. For example, in [1], improving the previous estimate $\ell_p = O((\log p)^2)$ of Lenstra [9] (see also [3, 6, 7]), the following bounds have been given:

$$\ell_p \leq \begin{cases} (\log p)^{463/252+o(1)} & \text{for all primes } p, \\ (\log p)^{5/3+o(1)} & \text{for almost all primes } p, \end{cases}$$

(where “almost all primes p ” means for all primes p but a set of relative density zero).

Here we use some results of [1], combined with the approach of Granville [4] to obtain new estimates on the cardinality of the sets

$$\begin{aligned} \mathcal{Q}_p(N) &= \{n \leq N : q_p(n) = 0\}, \\ \mathcal{R}_p(N) &= \{\ell \leq N : \ell \text{ prime, } q_p(\ell) = 0\}, \end{aligned}$$

which for small N improve that of [4]. We apply these improvements to study the sums

$$S_p = \sum_{n \in \mathcal{Q}_p(p)} \frac{\Lambda(n)}{n}$$

introduced by Ihara [7], where, as usual,

$$\Lambda(n) = \begin{cases} \log \ell, & \text{if } n \text{ is a power of a prime } \ell, \\ 0, & \text{otherwise,} \end{cases}$$

be the *von Mangoldt function*.

We note that in [7, Corollary 7], under the *Generalised Riemann Hypothesis*, the bound

$$S_p \leq 2 \log \log p + 2 + o(1) \tag{1}$$

as $p \rightarrow \infty$, has been obtained. Here we give an unconditional proof of a stronger bound.

Throughout the paper, the implied constants in the symbols ‘ O ’, and ‘ \ll ’ may occasionally depend on the real positive parameter α and are absolute otherwise (we recall that the notation $U \ll V$ is equivalent to $U = O(V)$).

2 Preparations

We recall that for any integers m and n with $\gcd(mn, p) = 1$ we have

$$q_p(mn) \equiv q_p(m) + q_p(n) \pmod{p}, \quad (2)$$

see, for example, [2, Equation (2)].

Let \mathcal{G}_p be the group of the p th power residues in the unit group $\mathbb{Z}_{p^2}^*$ of the residue ring \mathbb{Z}_{p^2} modulo p^2 .

Lemma 1. *For any $u \in \mathbb{Z}_{p^2}^*$ the conditions $q_p(u) = 0$ and $u \in \mathcal{G}_p$ are equivalent.*

Proof. Clearly $q_p(u) = 0$ for $u \in \mathbb{Z}_{p^2}^*$ is equivalent to $u^{p-1} \equiv 1 \pmod{p^2}$, which in turn is equivalent to $u \in \mathcal{G}_p$. \square

Let $T_p(K)$ be the number of $w \in [1, K]$ such that their residues modulo p^2 belong to \mathcal{G}_p . The following estimate follows immediately from [1, Equation (12)].

Lemma 2. *For any fixed*

$$\alpha > \frac{463}{252},$$

and

$$K \geq p^\alpha$$

we have

$$T_p(K) \ll K/p.$$

Let $\tau_s(n)$ be the number of representations of n as a product of s positive integers:

$$\tau_s(n) = \#\{(n_1, \dots, n_s) \in \mathbb{N}^s \mid n = n_1 n_2 \dots n_s\}.$$

We also need the following upper bound from [11]:

Lemma 3. *Uniformly over n and s we have*

$$\tau_s(n) \leq \exp \left(\frac{(\log n)(\log s)}{\log \log n} \left(1 + O \left(\frac{\log \log \log n + \log s}{\log \log n} \right) \right) \right).$$

In particular, we have:

Corollary 4. *If $s = (\log n)^{o(1)}$ then*

$$\tau_s(n) \leq n^{o(1)}.$$

as $n \rightarrow \infty$.

3 Distribution of vanishing Fermat quotients

Here we estimate the cardinality of the sets $\mathcal{Q}_p(N)$ and $\mathcal{R}_p(N)$. For large values of N , namely for $N \geq p^\alpha$ with $\alpha > 463/252$ such a bound is given by Lemma 2. However here we are mostly interested in small values of N .

We note that Granville [4] has given a bound on the cardinality of the set $\mathcal{R}_p(N)$. Namely, it is shown in [4] that for $u = 1, 2, \dots$

$$\#\mathcal{R}_p(p^{1/u}) \leq up^{1/2u}. \quad (3)$$

We note that the argument used in the proof of (3) can be used to estimate $\#\mathcal{R}_p(p^{1/u})$ for any $u \geq 1$.

We derive now upper bounds on $\#\mathcal{Q}_p(N)$ and $\#\mathcal{R}_p(N)$ that improve (3).

Theorem 5. *For any fixed*

$$\alpha > \frac{463}{252},$$

for $1 \leq u = (\log p)^{o(1)}$, where

$$u = \frac{\log p}{\log N},$$

we have

$$\#\mathcal{Q}_p(N) \ll uNp^{-(1+o(1))/\lceil \alpha u \rceil}.$$

as $p \rightarrow \infty$.

Proof. We put

$$s = \lceil \alpha u \rceil.$$

We consider $(\#\mathcal{Q}_p(N))^s$ products $n = n_1 \dots n_s$ where $(n_1, \dots, n_s) \in \mathcal{Q}_p(N)^s$. By (2) we see that

$$q_p(n) = q_p(n_1) \dots q_p(n_s) = 0.$$

Besides, using Corollary 4 we see that each $n \leq N^s < p^{\alpha+1}$ has at most

$$\tau_s(n) = p^{o(1)}$$

such representations. We also note that $N^s \geq p^\alpha$. Therefore, combining Lemmas 1 and 2, we derive

$$(\#\mathcal{Q}_p(N))^s \leq T_p(N^s)p^{o(1)} \leq N^s p^{-1+o(1)},$$

which implies the desired result. \square

Corollary 6. *If*

$$\frac{\log p}{\log N} = (\log p)^{o(1)} \quad \text{and} \quad \frac{\log p}{\log N} \rightarrow \infty$$

then

$$\#\mathcal{Q}_p(N) \leq N^{211/463+o(1)}$$

as $p \rightarrow \infty$.

For the set $\mathcal{R}_p(N)$ we have a bound in a wider range of u .

Theorem 7. *For any fixed*

$$\alpha > \frac{463}{252},$$

for $u \geq 1$, *where*

$$u = \frac{\log p}{\log N},$$

we have

$$\#\mathcal{R}_p(N) \ll u N p^{-1/\lceil \alpha u \rceil}$$

as $p \rightarrow \infty$.

Proof. The proof is the same as that of Theorem 5 except that instead of Corollary 4 we note that there are at most $s!$ products of s primes $\ell_1 \dots \ell_s$ that take the same value. So, we derive

$$(\#\mathcal{R}_p(N))^s \ll s! T_p(N^s) \ll s! N^s p^{-1},$$

and the result now follows. □

Corollary 8. *If* $N < p$ *and*

$$\frac{\log p}{\log N} \rightarrow \infty$$

then

$$\#\mathcal{R}_p(N) \leq N^{211/463+o(1)} \log p$$

as $p \rightarrow \infty$.

4 Ihara sums

First we consider approximations of S_p by partial sums

$$S_p(N) = \sum_{n \in \mathcal{Q}_p(N)} \frac{\Lambda(n)}{n}.$$

Theorem 9. *For $N = p^{o(1)}$ we have*

$$S_p = S_p(N) + O(N^{-252/463+o(1)} \log p)$$

as $p \rightarrow \infty$.

Proof. Clearly, we have

$$S_p - S_p(N) = \sum_{\substack{\ell > N \\ \ell \in \mathcal{R}_p(p)}} \frac{\log \ell}{\ell} + O(N^{-1} \log N). \quad (4)$$

We now see from Corollary 5 that for any

$$L < N^3$$

we have

$$\begin{aligned} \sum_{\substack{2L \geq \ell > L \\ \ell \in \mathcal{R}_p(p)}} \frac{\log \ell}{\ell} &\leq \frac{\log L}{L} \sum_{\ell \in \mathcal{R}_p(2L)} 1 \\ &\leq \frac{\log L}{L} L^{211/463+o(1)} \log p = L^{-252/463+o(1)} \log p. \end{aligned} \quad (5)$$

For

$$p \geq L > N^3$$

we choose

$$\alpha = \frac{463}{251}$$

and note that for $u \geq 1$ we have

$$\lceil \alpha u \rceil \leq \frac{3}{2} \alpha u.$$

Thus Theorem 7 implies the bound

$$\#\mathcal{R}_p(L) \ll L^{1-2/3\alpha} \log p \ll L^{2/3} \log p.$$

Hence in the above range, we have

$$\begin{aligned} \sum_{\substack{2L \geq \ell > L \\ \ell \in \mathcal{R}_p(p)}} \frac{\log \ell}{\ell} &\leq \frac{\log L}{L} \sum_{\ell \in \mathcal{R}_p(2L)} 1 \\ &\leq \frac{\log L}{L} L^{2/3} \log p = L^{-1/3+o(1)} \log p. \end{aligned} \tag{6}$$

Thus covering the range $[N, p]$ by dyadic intervals of the form $[L, 2L]$ and using the bounds (5), and (6) we derive

$$\sum_{\substack{\ell > N \\ \ell \in \mathcal{R}_p(p)}} \frac{\log \ell}{\ell} \leq N^{-252/463+o(1)} \log p,$$

which after the substitution in (4) implies the desired estimate. \square

Since by the Mertens formula (see, for example, [8, Equation (2.14)])

$$S_p(N) \leq \sum_{n \leq N} \frac{\Lambda(n)}{n} = \log N + O(1),$$

we derive from Theorem 9:

Corollary 10. *For $N = p^{o(1)}$ we have*

$$S_p \leq \log N + O(N^{-252/463+o(1)} \log p + 1)$$

as $p \rightarrow \infty$.

We now obtain an unconditional improvement of the conditional estimate (1).

Corollary 11. *We have*

$$S_p \leq (463/252 + o(1)) \log \log p$$

as $p \rightarrow \infty$.

Proof. Taking $N = \lceil (\log p)^\alpha \rceil$ with $\alpha > 463/252$ in the bound of Corollary 10 leads to the estimate

$$S_p \leq \alpha \log \log p + O(1).$$

Since α is arbitrary, the result now follows. \square

5 Index of some subfields of cyclotomic fields

We recall that the index $I(\mathbb{K})$ of an algebraic number field \mathbb{K} is the greatest common divisor of indexes $[\mathcal{O}_{\mathbb{K}} : \mathbb{Z}[\xi]]$ taken over all $\xi \in \mathcal{O}_{\mathbb{K}}$, where $\mathcal{O}_{\mathbb{K}}$ is the ring of integers of \mathbb{K} .

As in [7], we denote by I_p the index of the field \mathbb{K}_p , which is the unique cyclic extension of degree p over \mathbb{Q} that is contained in the cyclotomic field $\mathbb{Q}(\exp(2\pi i/p^2))$.

It has been shown in [7, Proposition 4 (i)] that under the Generalised Riemann Hypothesis the bound

$$\log I_p \leq (1 + o(1))p^2 \log \log p \quad (7)$$

holds as $p \rightarrow \infty$. Also [7, Proposition 5] gives an unconditional but weaker bound

$$\log I_p \leq (1/4 + o(1))p^2 \log p.$$

We use Corollary 11 to obtain an unconditional improvement of (7).

Theorem 12. *We have*

$$\log I_p \leq \left(\frac{463}{504} + o(1) \right) p^2 \log \log p$$

as $p \rightarrow \infty$.

Proof. By [7, Equation (2.4.1)] we have

$$\log I_p = \sum_{n \in \mathcal{Q}_p(p)} \alpha_p(n) \Lambda(n), \quad (8)$$

where

$$\alpha_p(n) = \left\lfloor \frac{p}{n} \right\rfloor \left(p - \frac{1}{2}n - \frac{1}{2} \left\lfloor \frac{p}{n} \right\rfloor n \right).$$

Since

$$\alpha_p(n) = \left\lfloor \frac{p}{n} \right\rfloor \left(p - \frac{1}{2}n \left(1 + \left\lfloor \frac{p}{n} \right\rfloor \right) \right) \leq \left\lfloor \frac{p}{n} \right\rfloor \frac{p}{2} \leq \frac{p^2}{2n},$$

we see from (8) that

$$\log I_p \leq \frac{p^2}{2} S_p.$$

Using Corollary 11, we conclude the proof. \square

One certainly expects that I_p is much smaller, than the bound given in Theorem 12, however no unconditional lower bound seems to be known (see [7, Proposition 4 (ii)] for a conditional estimate).

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